Nonlinear Behavior of Thin Columns Under a Parametrically Excited Load

M. Sunakawa* and K. Higuchi†
The University of Tokyo, Tokyo, Japan

The nonlinear behavior of a simply supported thin column in a parametrically unstable state is investigated analytically and experimentally. With regard to energy transfer, the variation of amplitude of the response is related in closed form to the phase difference between excitation and response, and the behavior of the column is examined in detail. It is also shown that the initial disturbances have little effect on the nonlinear response, contrary to the results of linear analysis. Furthermore, the response under comparatively large excitation load is presented. The experimental results show good agreement with the theoretical and numerical ones. It becomes clear that the behavior can be evaluated using a relatively simple nonlinear equation and the present theoretical results, to analyze such parametrically unstable phenomena as those described in the present paper.

I. Introduction

IN many previous investigations of parametric instability phenomena, attention has been paid primarily to defining the boundaries of the instability regions.[1-7] In optimum structural design, both the determination of the instability boundaries of structural components and the clarification of their behavior in and around the instability regions[8,9] are required. In the present paper, the nonlinear behavior of a simply supported thin column in the above-mentioned regions is investigated analytically and experimentally.

In theoretical and numerical analyses, the characteristics of beat as well as the behavior of the column in instability regions and the response at large values of excitation parameter are made clear. By paying special attention to the change in phase difference with time between excitation and response, the energy transfer and therefore beat phenomena are explained in a straightforward manner.

Some experimental investigations are carried out under a controlled excitation force. The experimental and analytical results show good agreement. Also it is pointed out that the behavior of the column in instability regions can be reproduced by numerical simulation. It then becomes clear that the behavior can be evaluated under pertinent considerations, using a relatively simple nonlinear equation and the appropriate theoretical results.

II. Theoretical Analysis

A. Governing Equation

A simply supported thin column (Beliaev's column) as shown in Fig. 1 is considered, where ℓ, EI, m, M, and $P(t)=P_0+P_t\cos\theta t$ are the length, flexural rigidity, line mass, concentrated mass, and axial load of the column, respectively. After separating the dependent variable (transverse displacement) into the time and spatial coordinates, the Galerkin method is applied. The governing equation of the problem[10] is then given by

$$\ddot{f}+2\epsilon\dot{f}+\Omega^2(1-2\mu\cos\theta t)f+2\kappa f(f\ddot{f}+\dot{f}^2)+\gamma f^3=0 \qquad (1)$$

where f is the transverse displacement at the midpoint of the column, ϵ the linear damping coefficients of the total system, $\Omega=\omega(1-P_0/P_b)^{1/2}$ the fundamental angular frequency of transverse vibration of the column subjected to axial load, ω the fundamental angular frequency of transverse vibration of the column, $\mu=P_t/[2(P_b-P_0)]$ the excitation parameter, P_0 the constant component of the axial load, P_t the amplitude of sinusoidal component of the axial load, P_b the static buckling load of the column, θ the angular frequency of excitation, $\kappa=[\pi^4/(4\ell^2)][1/3-3/(8\pi^2)+M/(m\ell)]$ the nonlinear inertial coefficient, $\gamma=\pi^2\omega^2/(8\ell^2)$ the nonlinear elastic coefficient, and $(\dot{\ })$ denotes a derivative with respect to time.

B. Analytical Results

At first, the maximum amplitude of beat is obtained. When the response shows the maximum amplitude due to beat, the time derivative of the amplitude of beat (envelope of the vibration amplitude) becomes temporarily very small (ap-

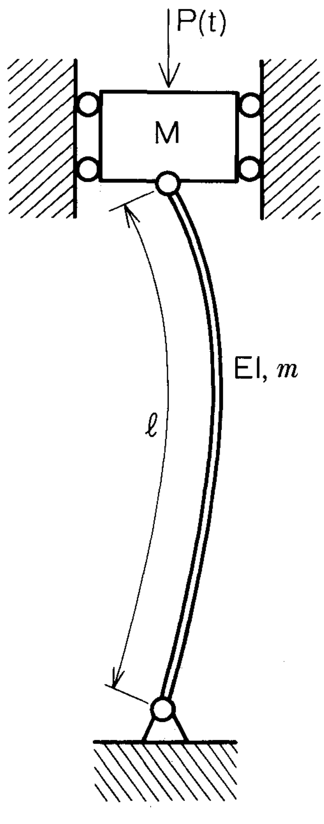


Fig. 1 Analytical model (Beliaev's column).

Presented as Paper 83-0862 at the AIAA/ASME/ASCE/AHS 24th Structures, Structural Dynamics and Materials Conference, Lake Tahoe, Nev., May 2-4, 1983; received Aug. 3, 1983; revision received Nov. 20, 1983.

*Professor, Institute of Interdisciplinary Research, Faculty of Engineering. Member AIAA.

†Graduate Student, Division of Engineering; currently Research Associate, Department of Industrial Machinery Engineering, Faculty of Science and Engineering, Tokyo Denki University, Hatoyama, Japan.

proximately zero). It is reasonable therefore to assume that the solution $f(t)$ has the following form:

$$f(t)=A_0\sin(\theta t/2+\phi) \tag{2}$$

where, A_0 and $\phi(t)$ are the maximum amplitude of beat and phase difference (gradually varying variable), respectively. Disregarding higher harmonics and the second time derivative of ϕ, the next approximation is obtained.

$$f(f\ddot{f}+\dot{f}^2)\simeq -\tfrac{2}{3}[(\theta/2)+\dot{\phi}]^2 f^3 \tag{3}$$

This equation means that the nonlinear inertia is replaced by the equivalent nonlinear stiffness,[11] that is, a softening spring that responds to the angular frequency $(\theta/2+\dot{\phi})$. Then Eq. (1) is reduced to

$$\ddot{f}+2\epsilon\dot{f}+\Omega^2(1-2\mu\cos\theta t)f+\left[\gamma-\frac{4}{3}\kappa\left(\frac{\theta}{2}+\dot{\phi}\right)^2\right]f^3=0 \tag{4}$$

After application of the Kryloff-Bogoliuboff method[12] to Eq. (4) and letting $\dot{A}=0$, the next solution is finally obtained.

$$A_0^2=\frac{1-n^2N^2+[\mu^2-(4\epsilon n^2/\theta)^2]^{1/2}}{n^2N^2\kappa-\tfrac{3}{4}(\gamma/\Omega^2)} \tag{5}$$

where $n=\theta/(2\Omega)$, and $N^2=1+4\dot{\phi}/\theta\simeq(1+2\dot{\phi}/\theta)^2$.

Equation (5) gives a closed-form relationship between the maximum amplitude A_0 and the time derivative of phase difference $\dot{\phi}$. The term $(\theta/2+\dot{\phi})$ is an apparent angular frequency of the nonstationary response and N is a modification factor of n; $N=1$ corresponds to the so-called stationary amplitude.

The results of direct numerical integration of Eq. (1) agree well with those of Eq. (5), as shown quantitatively in Table 1 and qualitatively in Fig. 2. That is, Eq. (5) gives an excellent approximation. It is shown in Fig. 2 that the time of local extrema and the period of the variation of ϕ coincide well with those of beat.

Figure 3 shows the relationship between the excitation and dissipation energies in the E-μ-ϕ space. The excitation energy

$$E_\mu=\mu\Omega^2\pi A^2\sin2\phi \tag{6}$$

and the dissipation energy

$$E_\epsilon=\epsilon\theta\pi A^2 \tag{7}$$

are the nonconservative terms in the vibration energy, which is obtained by the energy integration of Eq. (1) under the assumption of fundamental harmonic stationary response Eq. (2). Figure 3 explains qualitatively the variation of stability characteristics as a function of energy transfer and also intuitively the relationship between the fluctuation in time of phase difference and the variation of amplitude in a beat. The hatched area in the figure corresponds to unstable solutions of the linear analyses where the phase difference is regarded as a constant. Periodic solutions corresponding to the stability boundary have the phase difference

$$\sin2\phi=\epsilon\theta/(\mu\Omega^2) \tag{8}$$

which is obtained from the condition $E_\mu=E_\epsilon$.

In the use of the direct numerical integration method, the initial condition plays an important role. However, the initial

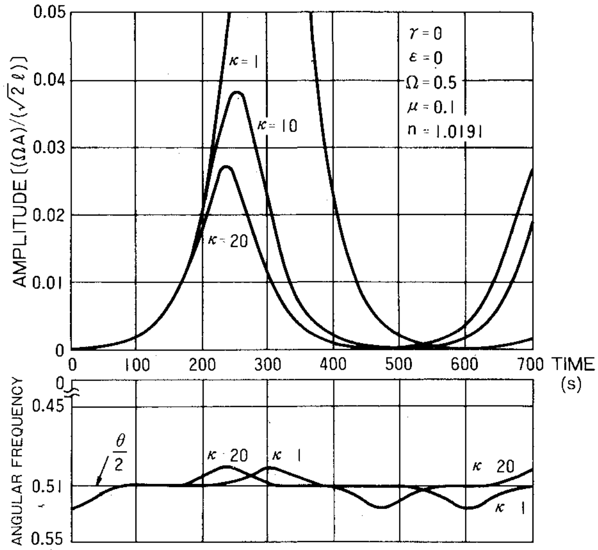


Fig. 2 Change of amplitude and angular frequency with time.

Table 1 Stationary amplitude, and maximum amplitude and frequency where $\dot{\phi}$ has been taken into account

κ	ϵ	μ	n	(A) A_s	(B) A_c	(C)	(D)
1	0.0			0.243	0.345	0.495	0.496
			1.0191		0.107	0.496	0.497
		0.1		0.077	0.102	0.498	0.499
	0.001				0.099	0.500	0.501
10			1.0393	0.048	0.059	0.515	0.515
					0.056	0.517	0.516
					0.094	0.544	0.546
	0.01	0.3	1.1103	0.072	0.084	0.551	0.551
					0.079	0.553	0.553
	0.0		1.0191	0.055	0.077	0.495	0.496
20		0.1			0.044	0.514	0.515
	0.001		1.0393	0.030	0.042	0.515	0.515
					0.039	0.516	0.516

N.B. (A): stationary amplitude evaluated by Eq. (5) when $N=1$ (A_s).
(B): maximum amplitude obtained by the numerical integration of Eq. (1) (A_c).
(C): angular frequency when maximum amplitude shown in (B) appears.
(D): angular frequency obtained by substituting maximum amplitude shown in (B) to Eq. (5); $(\theta/2+\dot{\phi})$.
($\gamma=0.0$, $\Omega=0.5$).

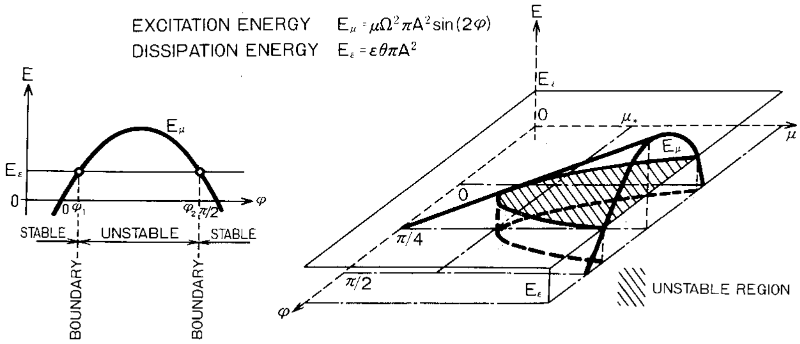


Fig. 3 Variation of principal instability region with energy transfer in parameter space.

disturbances, regardless of their type, have little effect on the nonlinear response, contrary to the results of linear analysis which indicate the beat in a stable region is sensitive to the initial disturbances.

The behavior of the column under comparatively large excitation parameters is discussed qualitatively. In Figs. 4-6, solid lines indicate the calculated transverse vibrations and broken ones the excitations. The composition of the higher harmonics around the upper boundary of the unstable region differs from that around the lower one, and this difference has a feature common in every unstable region. Some examples are shown in Fig. 4. In the case of $\mu > 0.5$, the absolute value of the excitation load $P_0 + P_t$ is larger than the static buckling load P_b. As shown in Fig. 5, in the case where the boundaries of neighboring unstable regions become closer, the response is influenced by both regions, which, in turn, makes Hill's assumption[13] invalid. When μ becomes much larger, as shown in Fig. 6, the column shows behavior something like an impulsive buckling, depending on the combination of n and μ.

III. Experimental Investigation

A. Experimental Procedure

The response of a simply supported thin column to a parametric compressive load has been observed. The applied load $P(t)$ is composed of constant and time-dependent components, that is, $P(t) = P_0 + P_t \cos\theta t$; the magnitude of P_0 is about one half of that of P_b, and in some cases $P(t)$ exceeds P_b periodically. The overall setup of the experiment is shown schematically in Fig. 7.

The devices are set up as shown in Fig. 8. The thin column 7 is excited at the movable lower end through an attachment jig 10 by an exciter 11 . The output of excitation force at the exciting point, measured by the lightweight force transducer 12 , is fed back to the exciter controller in order to keep P_t constant. P_0 is applied to the column by means of coil springs 6 . The axial displacement at the exciting point and the transverse one at the midpoint of the column are picked up by noncontact electro-optical devices 8 and 13 , respectively.

The fluctuation in P_0 results from the axial extensional vibration of the springs caused by the transverse vibration of the column. Thus, in order to limit the fluctuation within 1% of P_0, long springs with small stiffness have been used, which are also effective in preventing the nonlinear elastic coefficient from becoming unnecessarily large. Attachment devices were designed to be as light as possible, because a large moving mass makes the nonlinear inertial coefficient too large.

Special attention has been paid to realize desired end conditions on the column. By providing circular rods on both ends of the column (Fig. 9 and Table 2) and miniature ball bearings in attachment jigs, simply supported end conditions have been given to the column.

All of the experimental data were stored in a data recorder and then reproduced on electromagnetic oscillograph paper.

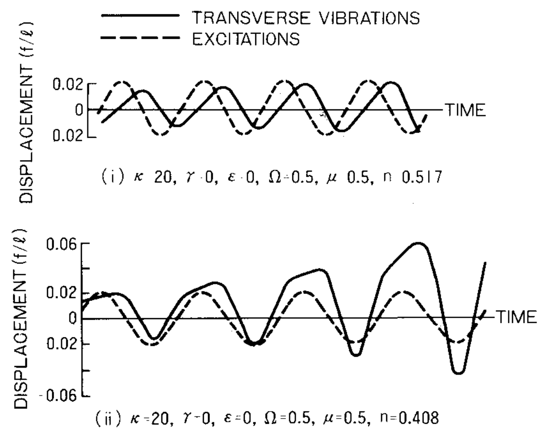


Fig. 4 Characterisitics of response below buckling load, $\mu = 0.5$.

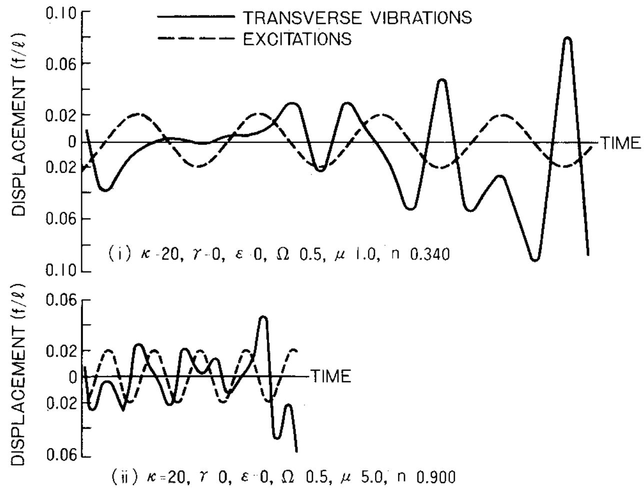


Fig. 5 Characteristics of response near buckling load, $\mu = 1.0, 5.0$.

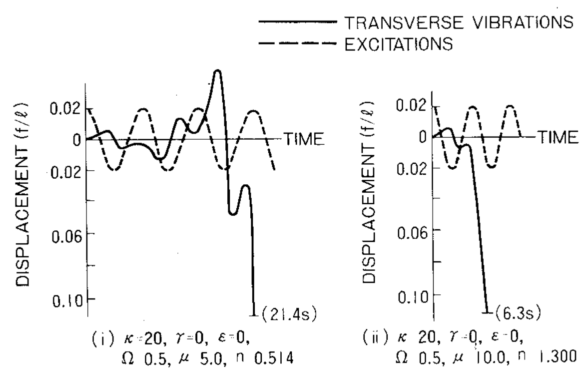


Fig. 6 Characteristics of response above buckling load, $\mu = 5.0, 10.0$.

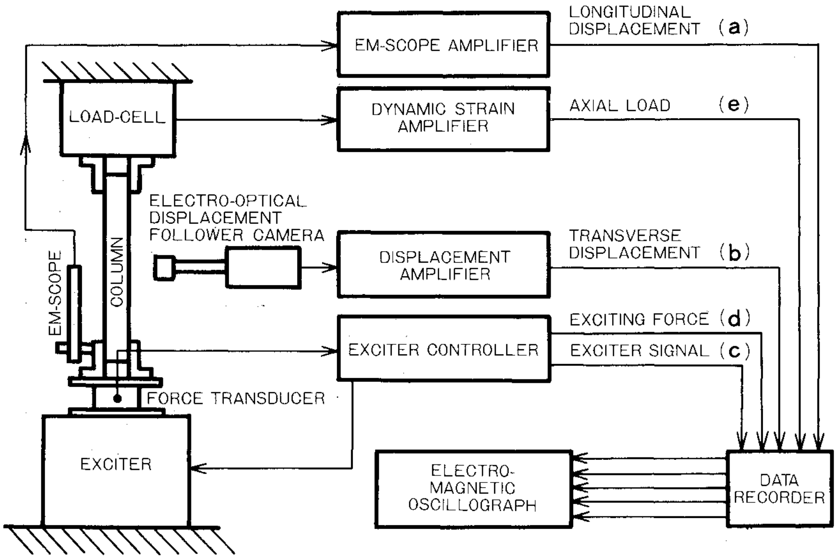


Fig. 7 Overall test setup.

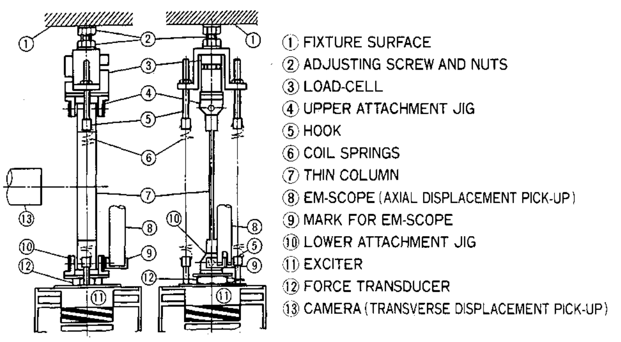


Fig. 8 Experimental devices.

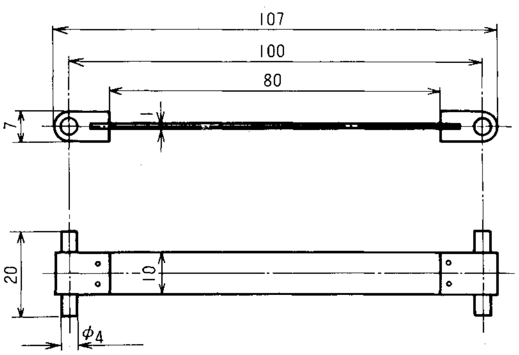


Fig. 9 Column geometry.

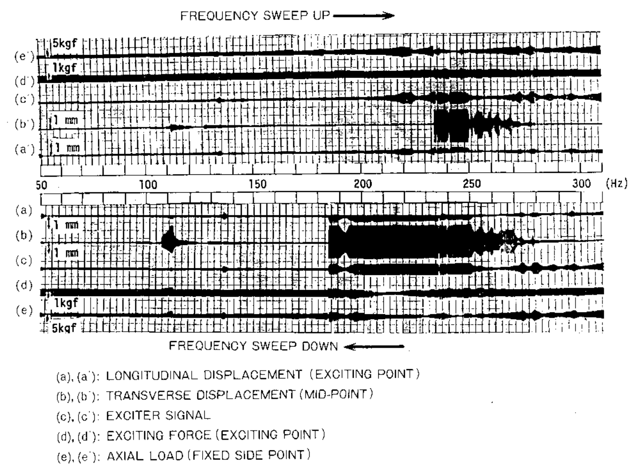


Fig. 10 Response of column to swept load.

Table 2 Thin column

Dimensions	100 × 10 × 1 mm
Material	5052 (Al alloy)
Young's modulus	7000 kg/mm^2
Density	2.7 g/cm^3
Static buckling load P_b	5.76 kg
Fundamental natural frequency of transverse vibration $\omega/(2\pi)$	228.5 Hz

B. Results and Discussion

Figure 10 shows the response of the column in the case where the excitation frequency is swept up and down. From this figure, it can be seen that the present column system has a nonlinear stiffness of the softening type, the parametric resonance is sensitive to the change in excitation frequency, and there exist beat phenomena around boundaries of instability regions which are different types of beat from that discussed in Sec. IIB. The jump phenomena are observed clearly in both the principal and subresonance regions. The amplitude in the subresonance region is asymmetric.

In Fig. 11, it can be seen that the phase difference between excitation and response in a beat changes with time, which shows the validity of the assumption made in the theoretical analysis.

The experimental results are evaluated quantitatively based on the following equation.

$$\ddot{f}+2\epsilon\dot{f}+\Omega^2(1-2\mu\cos\theta t)f+2\kappa f(f\ddot{f}+\dot{f}^2)+\gamma f^3+2\epsilon_N f^2\dot{f}=0 \tag{9}$$

where ϵ_N is the nonlinear damping coefficient.

Equation (9) has three nonlinear parameters: κ, γ, and ϵ_N. The values are: $\kappa=4.47\times10^{-1}$ mm^{-2}, $\gamma=2.14\times10^4$ s^{-2}mm^{-2}, $\zeta=\epsilon/\Omega=2.0\times10^{-2}$, and $\zeta_N=\epsilon_N/\Omega=4.94\times10^{-1}$ mm^{-2}, where κ and γ have been calculated theoretically and ϵ and ϵ_N have been measured experimentally. From the experimental results, it was found that the ratios of the amplitudes of the second and third modes to that of the first mode were 0.006 and 0.096, respectively. Hence, in the formulation, higher order terms were neglected.

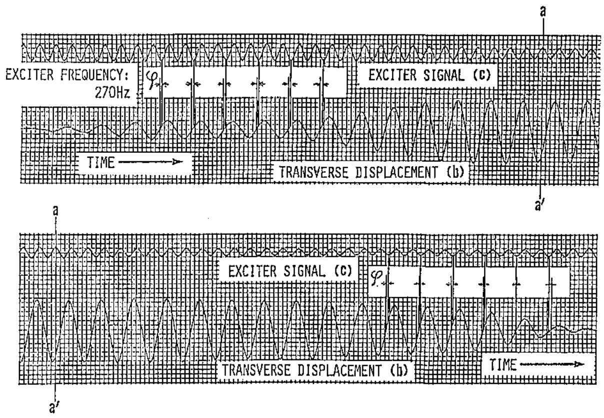
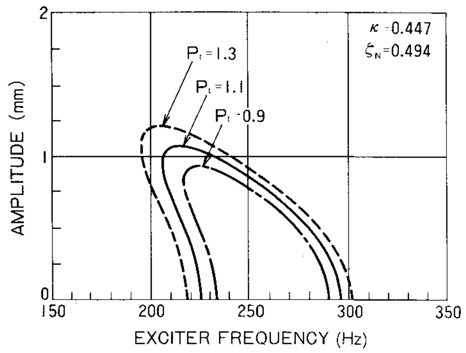
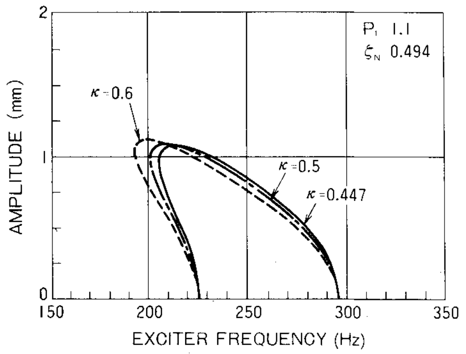


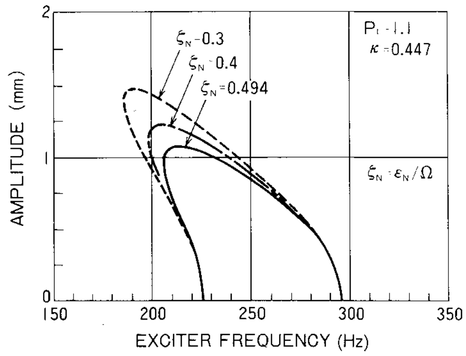
Fig. 11 Phase difference between excitation and response.



a) Effect of variation of P_t.

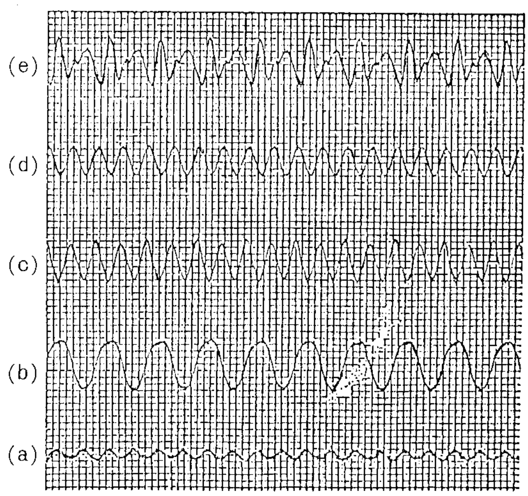


b) Effect of variation of κ.



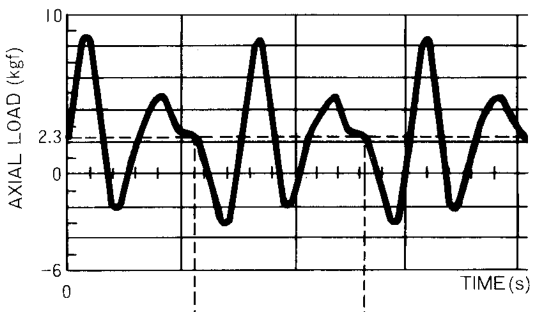
c) Effect of variation of ϵ_N.

Fig. 12 Response curves of parametric excitation.

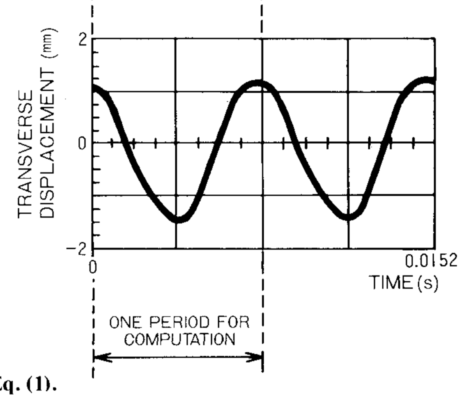


(a): LONGITUDINAL DISPLACEMENT (EXCITING POINT)
(b): TRANSVERSE DISPLACEMENT (MID-POINT)
(c): ACCELERATION (EXCITING POINT)
(d): EXCITER SIGNAL ; 260Hz
(e): AXIAL LOAD (FIXED SIDE POINT)

a) Experimental results.



b) Applied load.



c) Simulation by Eq. (1).

Fig. 13 Results of numerical simulation.

Table 3 Comparison of amplitudes of transverse vibration

Excitation frequency, Hz	Amplitude of sinusoidal component of load, kg	Amplitude of transverse vibration, mm Experiment	Theory
271.9	2.33	1.02[a]	0.96
268.9	2.22	1.00[a]	0.97
258.2	1.00	0.71	0.72
256.3	1.56	1.20[a]	0.95
194.4	1.33	1.22	1.19
192.2	1.72	1.23	1.46

[a] Maximum amplitude due to beat.

The variation of the resonance curve with changes in amplitude of the sinusoidal component of axial load P_t, nonlinear inertial coefficient κ, and nonlinear damping coefficient ϵ_N have also been shown in Fig. 12. The solid lines in the figures have been obtained using the values of parameters presented in the preceding paragraph. The jump frequency of the solid line does not necessarily agree exactly with the results shown in Fig. 10. This would seem to be attributed to the simplification that only the lowest nonlinearities have been considered in Eq. (9), and also to the error in evaluating the parameters. It is difficult to keep P_t constant throughout resonances. Also, the values of the nonlinear coefficients have a great influence on the theoretical results.

Nevertheless, the experimental and theoretical values of amplitude have agreed satisfactorily with each other, as shown in Table 3. The maximum amplitudes in beat are larger than those calculated under the assumption of stationary vibration as stated in Sec. IIB.

C. Numerical Simulation

As an example, a result of numerical simulation is compared with experimental data in Fig. 13; these data are the result of special experiments where the excitation load was not controlled. The record of acceleration, (c), Fig. 13a, shows that the excitation was sinusoidal, but that of axial load, (e), was deformed as a result of transverse vibration, because the excitation load was not controlled in this special case. Using the digital load pattern [Fig. 13b obtained from (e) in Fig. 13a], Eq. (1) was integrated numerically.

The values of Ω, μ, κ, and γ were calculated theoretically, while those of θ, P_0, and P_t were the ones under the actual experiment. The value of ϵ was obtained from the condition for the stationary vibration, that is, the case of $N=1$ in Eq. (5). The initial phase angle ϕ was set according to Eq. (8). The value of θ, Ω, μ, ϵ, κ, and γ used in the numerical simulation are as follows: $\theta/(2\pi)=260$ Hz, $\Omega=1.11\times10^3$ rad/s, $\mu=0.72$, $\epsilon=1.94\times10^2$ s^{-1}, ($\eta=2\epsilon/\Omega=0.348$), $\kappa=5.21\times10^{-1}$ mm^{-2}, and $\gamma=1.99\times10^4$ s^{-2}mm^{-2}.

The simulated vibration mode in Fig. 13c agrees well with the experimentally obtained one, (b) in Fig. 13a. This means that the nonlinear behavior of the column can be simulated under pertinent considerations, even in unstable regions, by using Eq. (1) and the results of the present theoretical analysis.

IV. Concluding Remarks

The nonlinear behavior of a simply supported thin column under parametrically excited load has been investigated analytically and experimentally. By reducing the effect of longitudinal inertia to that of nonlinear transversal restoring force, the nonlinear equation of motion has been solved by the Kryloff-Bogoliuboff method.

The characteristics of beat have been made clear in both the stable and unstable regions, where special attention has been paid to the change in phase difference with time between excitation and response. The influence of initial conditions upon the stability of the column and the behavior at large values of excitation parameter have also been discussed.

The results of experiments carried out under the controlled excitation load have shown good agreement with the analytical ones, and it has been pointed out that the behavior of the column can be reporduced by numerical simulation, even in instability regions. Also, it becomes clear that the behavior can be evaulated under pertinent considerations, by the use of a relatively simple nonlinear equation and the present theoretical results.

References

[1] Uchida, I. and Sezawa, K., "Dynamic Stability of a Column under Periodic Longitudinal Forces," Aeronautical Research Institute, Tokyo Imperial University, Report 193, 1940.

[2] Hsu, C. S., "On the Parametric Excitation of a Dynamic System Having Multiple Degrees of Freedom," *Journal of Applied Mechanics*, Vol. 30, Sept. 1963, pp. 367-372.

[3] Burney, S. Z. H. and Jeager, L. G., "A Method of Determining the Regions of a Column by a Numerical Method Approach," *Journal of Sound and Vibration*, Vol. 15, 1971, pp. 75-91.

[4] Fu, F. C. L. and Nemat-Nasser, S., "Stability of Solution of Systems of Linear Differential Equations with Harmonic Coefficients," *AIAA Journal*, Vol. 10, Jan. 1972, pp. 30-36.

[5] Iwatsubo, T., Saigo, M., and Sugiyama, Y., "Parametric Instability of Clamped-Clamped and Clamped-Simply Supported Columns under Periodic Axial Load, *Journal of Sound and Vibration*, Vol. 30, 1973, pp. 65-77.

[6] Hsu, C. S., "On Nonlinear Parametric Excitation Problems," *Advances in Applied Mechanics*, Vol. 17, 1977, pp. 245-301.

[7] Ibrahim, R. M., "Parametric Vibration, Part I-V," *Shock and Vibration Digest*, Vol. 10, 1978.

[8] Somerset, J. H. and Evan-Iwanowski, R. M., "Experiments on Parametric Instability of Columns," *Developments in Theoretical and Applied Mechanics*, Proceedings of 2nd SECTAM, 1964, pp. 503-525.

[9] Iwatsubo, T., Sugiyama, Y., and Ishihara, K., "Stability and Nonstationary Vibration of Columns under Periodic Loads," *Journal of Sound and Vibration*, Vol. 23, 1972, pp. 245-257.

[10] Bolotin, V. V., *Dynamic Stability of Elastic Systems*, Holden-Day, 1964.

[11] Higuchi, K. and Sunakawa, M., "Behavior of Thin Columns in Parametric Instability Regions," *Journal of the Japan Society for Aeronautical and Space Sciences*, Vol. 31, Jan. 1983, pp. 53-63 (in Japanese).

[12] Kryloff, N. and Bogoliuboff, N., *Introduction to Non-Linear Mechanics*, Princeton University Press, Princeton, N. J., 1950.

[13] Meirovich, L., *Methods of Analytical Dynamics*, McGraw-Hill Book Co., New York, 1970.

[14] Higuchi, K., "Nonlinear Analyses of Structural Components Subjected to Parametric Excitation," Dissertation, The University of Tokyo, 1983 (in Japanese).